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Quasisimilar operators with different spectra

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1. Introduction. Let $\mathscr{L}(\mathfrak{X})$ be the Banach algebra of all (bounded linear) operators acting on the complex Banach space $\mathfrak{X}$. $T\in\mathscr{L}(\mathfrak{X})$ and $A\in\mathscr{L}(\mathfrak{Y})$ are called *quasisimilar* (q.s.) provided there exist quasi-invertible continuous linear maps $X\colon \mathfrak{Y}\to\mathfrak{X}$ and $Y\colon \mathfrak{X}\to\mathfrak{Y}$ such that $TX=XA$ and $YT=AY$ (X is quasi-invertible if $\operatorname{Ker} X=\{0\}$ and $\operatorname{Ran} X$ is dense in $\mathfrak{X}$; [48]).

As in [38], the four (weakly closed identity containing) subalgebras naturally associated with $T\in\mathscr{L}(\mathfrak{X})$ will be denoted by $\mathscr{A}(T)$, $\mathscr{A}^a(T)$, $\mathscr{A}'(T)$ and $\mathscr{A}''(T)$ (the algebra generated by the polynomials in T, the algebra generated by the rational functions of T with poles outside the spectrum $\sigma(T)$ of T, the commutant and the double commutant of T, resp.). Then $\mathscr{A}(T)\subset\mathscr{A}^a(T)\subset\mathscr{A}''(T)\subset\mathscr{A}'(T)$ and the corresponding invariant subspace lattices satisfy the reverse inclusions: $\operatorname{Lat} T\overset{\text{def}}{=}\operatorname{Lat}\mathscr{A}(T)\supset\operatorname{Lat}\mathscr{A}^a(T)\supset\operatorname{Lat}\mathscr{A}''(T)\supset\operatorname{Lat}\mathscr{A}'(T)$. (These are called the lattices of invariant, analytically invariant, bi-invariant and hyperinvariant subspaces, resp. As usual, *subspace* will denote a *closed linear manifold* of $\mathfrak{X}$.)

Quasisimilarity was first studied by B. Sz.-Nagy and C. Foiaş ([48]; see also [17]) in connection with the invariant subspace problem in Hilbert spaces; namely, if A is q.s. to T, and T has a non-trivial hyperinvariant subspace, then so does A ([17; 39; 41]). A and T need not have the same spectrum ([48]); however, $\sigma(A)\cap\sigma(T)$ cannot be empty ([39]). Furthermore, every component of $\sigma(A)$ $(\sigma(T))$ intersects $\sigma(T)$ $(\sigma(A)$, resp.; [32]).

Several results scattered through the literature assert that, under suitable restrictions on T or A or both, $\sigma(A)$ actually contains $\sigma(T)$ or coincides with it ([9; 11; 39]) and there also exist examples of q.s. operators with different spectra ([39; 48]; see also Section 2, below).

This article is primarily concerned with the following questions:

(1) Under what conditions on T does "A is q.s. to T" imply "A is similar to T"?

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(2) Under what conditions on T does "A is q.s. to T" imply $\sigma(A)=\sigma(T)$?

(3) When can we assert that $\sigma(A)$ is strictly larger (or strictly smaller) than $\sigma(T)$ for some A q.s. to T?

It is completely apparent that if T satisfies (1), then it also satisfies (2). On the other hand, two q.s. nilpotent operators with infinite dimensional range acting on a separable Hilbert space need not be similar ([3]; see also [18; 36]), so that a T satisfying (2) need not satisfy (1).

In [2], C. Apostol proved that A is q.s. to a normal operator if and only if Lat A contains a countable *basic system of subspaces* $\{\mathfrak{R}_n\}_1^m$ $(1\leqq m\leqq\infty)$ such that $A|\mathfrak{R}_n$ (A restricted to $\mathfrak{R}_n$) is similar to a normal operator for every n. (A countable family $\{\mathfrak{X}_n\}_1^m$ of subspaces of the Banach space $\mathfrak{X}$ is called basic if the subspaces $\mathfrak{X}_n$ and $\mathfrak{X}_n'=\bigvee_{k\neq n}\mathfrak{X}_k$ are complementary for every n and $\bigcap_1^m\mathfrak{X}_n'=\{0\}$; [2]). In Section 2 it will be shown that, under suitable (very general) conditions, an operator T having a *denumerable* basic system of invariant subspaces is q.s. to operators A and B such that either $\sigma(A)$ is strictly smaller than $\sigma(T)$, or $\sigma(B)$ is strictly larger than $\sigma(T)$, or both. To the best of the author's knowledge, this is the only known way to produce q.s. operators with different spectra. Recently, L. A. Fialkow showed that two q.s. non-invertible injective bilateral weighted shifts need not be similar; however, they necessarily have the same spectrum and this spectrum can be a disc of positive radius. Since Fialkow's operators do not admit any non-trivial pair of complementary invariant subspaces (see [22]), they add some extra support to the following

Conjecture 1. *Assume that* Lat T *does not contain any denumerable basic system of subspaces. Then* $\sigma(A)=\sigma(T)$ *for every* A *q.s. to* T.

The *strict multiplicity* $\bar\mu(\mathscr{A})$ of a subalgebra $\mathscr{A}$ of $\mathscr{L}(\mathfrak{X})$ is defined as the infimum of card(Γ), taken over all the subsets Γ of $\mathfrak{X}$ such that $\mathfrak{X}=\left\{\sum_1^n A_jx_j\colon A_j\in\mathscr{A},\ x_j\in\Gamma,\ n=1,2,\ldots\right\}$. If Γ can be taken equal to the singleton $\{x_0\}$, then $\mathscr{A}$ is called a *strictly cyclic algebra* and x_0 is called a *strictly cyclic vector* for $\mathscr{A}$. According to [28, Theorem 8], if $\bar\mu[\mathscr{A}''(T)]<\infty$, then T satisfies (1). The main part of this paper is devoted to exploit this result and the constructions in [6] in order to show the existence and/or the density of operators satisfying certain properties related with quasisimilarity and an approximation problem, acting on a *complex separable infinite dimensional Hilbert space* $\mathfrak{H}$ (throughout this paper $\mathfrak{H}$ will always denote a space of this type).

Recall that $T\in\mathscr{L}(\mathfrak{H})$ is *biquasitriangular* (*BQT*) if ind$(\lambda-T)=0$, whenever $\lambda-T$ is a semi-Fredholm operator ([4]). C. Foiaş, C. Pearcy and D. Voiculescu [19] proved that for every $T\in\mathscr{L}(\mathfrak{H})$ and $\varepsilon>0$, there exists $T_\varepsilon\in\mathscr{L}(\mathfrak{H})$ such that

$\|T-T_\varepsilon\|<\varepsilon$, $T-T_\varepsilon\in\mathscr{K}$ (the ideal of compact operators), $T_\varepsilon=\text{norm-lim } U_n T U_n^*$ for a suitable sequence $\{U_n\}$ of unitary operators, Lat T_ε contains a denumerable family of pairwise orthogonal subspaces and T_ε is q.s. to a BQT operator ($T_\varepsilon\in(BQT)_{qs}$, in the notation of [19]). This strong result suggested to the authors of that article the following question

$$Is\ (BQT)_{qs}=\mathscr{L}(\mathfrak{H})?$$

The answer is no. Indeed, the following sets are (norm-)dense in $\mathscr{L}(\mathfrak{X})$:

(A) $=\{T$: T is q.s. to some $A\in(BQT)$ with $\sigma(A)=\sigma(T)\}$ [19];

(B) $=\{T$: T is q.s. to some $A\in(BQT)$ with $\sigma(A)\supset\sigma(T)$, $\sigma(A)\neq\sigma(T)\}$;

(C) $=\{T$: T is q.s. to some $A\in(BQT)$ with $\sigma(A)\subset\sigma(T)$, $\sigma(A)\neq\sigma(T)\}$;

(D) $=\{T$: T is similar to $A\oplus B$, $\bar{\mu}[\mathscr{A}''(A)]=\bar{\mu}[\mathscr{A}''(B^*)]=1$, $\sigma(A)\cap\sigma(B)=\emptyset$, λ_A-A and λ_B-B^* are semi-Fredholm operators of index $-\infty$ for suitably chosen points $\lambda_A,\lambda_B\in\mathbf{C}\}$.

Clearly, for every such T and every L q.s. to T, L is actually similar to T and it has the same spectrum as T. Therefore, (D) $\subset\{T$: T satisfies (1)$\}\setminus(BQT)_{qs}$.

$(E)_{mn}=\{T$: T, A and B are as in (D), except that $\bar{\mu}[\mathscr{A}''(A)]=m$ and $\bar{\mu}[\mathscr{A}''(B^*)]=n\}$ (for every m,n such that $m,n=1,2,\dots$ or c, the power of the continuum);

(F) $=\{T$: $\sigma(T)=\sigma(L)$ for every L q.s. to T, but $\mathscr{S}(T)\neq\mathscr{S}_{qs}(T)\}$, where $\mathscr{S}(T)$ ($\mathscr{S}_{qs}(T)$, resp.) $=\{A\in\mathscr{L}(\mathfrak{H})$: $A=WTW^{-1}$ for some invertible $W\in\mathscr{L}(\mathfrak{H})$ (A is q.s. to T, resp.)$\}$.

Recall that $\mathscr{A}\subset\mathscr{L}(\mathfrak{X})$ is a *reflexive algebra* if $\mathscr{A}=\text{Alg Lat}\,\mathscr{A}$, where $\text{Alg}\,\Sigma=\{A\in\mathscr{L}(\mathfrak{X})$: Lat $A\supset\Sigma\}$ (Σ = any family of subspaces of $\mathfrak{X}$). $T\in\mathscr{L}(\mathfrak{X})$ is called reflexive if $\mathscr{A}(T)$ is. The following results are "in the air": The sets

(G) $=\{T$: T is reflexive$\}$;

(H) $=\{T$: $\mathscr{A}^a(T)$ is reflexive$\}$;

(I) $=\{T$: $\mathscr{A}''(T)$ is reflexive$\}$;

(J) $=\{T$: $\mathscr{A}'(T)$ is reflexive$\}$,

as well as their complements in $\mathscr{L}(\mathfrak{H})$, are dense in $\mathscr{L}(\mathfrak{H})$.

There are at least two different extensions of the notion of similarity related with approximation problems: A and T are *asymptotically similar* if their similarity orbits have the same closure (i.e., $\mathscr{S}(A)^-=\mathscr{S}(T)^-$; [7; 33]). They are *approximately similar* if $A=\text{norm-lim } W_n T W_n^{-1}$ for a sequence $\{W_n\}$ of invertible operators with $\sup\|W_n\|\,\|W_n^{-1}\|<\infty$ ([24]). Since asymptotic similarity (and, a fortiori, approximate similarity) preserves the spectrum and every part of it (see [33]), it will not be difficult to conclude from the results and examples of this article and the results of [7; 8; 33; 34; 35] that, in general, $\mathscr{S}(T)$ is a proper subset of $\mathscr{S}_{ap}(T)\cap\cap\mathscr{S}_{qs}(T)$ ($\mathscr{S}_{ap}(T)=\{A$: A is approximately similar to $T\}\subset\mathscr{S}(T)^-$) and the equality $\mathscr{S}(T)=\mathscr{S}(T)^-$, $T\in\mathscr{L}(\mathfrak{H})$, implies that T is similar to a normal operator with a finite spectrum and therefore $\mathscr{S}(T)=\mathscr{S}_{qs}(T)=\mathscr{S}_{ap}(T)=\mathscr{S}(T)^-$ (this is false for arbitrary Banach spaces; see [7; 35]); however, the equality $\mathscr{S}(T)=\mathscr{S}_{qs}(T)$ does not imply $\mathscr{S}(T)=\mathscr{S}(T)^-$ (even for Hilbert spaces; [28; 35]. Since approximate similarity preserves every Schatten p-ideal and asymptotic similarity does not preserve them, it is immediate that these two notions are different; see [33; 46] for details).

In [24], D. W. HADWIN defined the *approximate double commutant* of $T\in\mathscr{L}(\mathfrak{H})$ by $\operatorname{appr}(T)''=\{L\in\mathscr{L}(\mathfrak{H})\colon \|LA_n-A_nL\|\to 0\ (n\to\infty)$ whenever $\{A_n\}$ is a *bounded* sequence such that $\|TA_n-A_nT\|\to 0\ (n\to\infty)\}$. He proved that $\operatorname{appr}(T)''\subset\mathscr{A}''(T)\cap C^*(T)$ (where $C^*(T)$ denotes the C^*-algebra generated by T) and conjectured ([24, Conjecture 2.5]) that $\operatorname{appr}(T)''=\mathscr{A}''(T)$ if and only if T is algebraic. This conjecture is false. Indeed, $(K)=\{T\colon \operatorname{appr}(T)''=\mathscr{A}''(T)\}$, as well as its complement, is dense in $\mathscr{L}(\mathfrak{H})$.

The interested reader will have no trouble to prove the density in $\mathscr{L}(\mathfrak{H})$ of new different classes of operators somehow related with $(A)-(K)$.

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2. **Operators quasisimilar to orthogonal direct sums.** Given a family $\{\mathscr{X}_n\}$ of Banach spaces, let $\mathscr{Y}=\bigoplus_1^\infty \mathscr{X}_n$ denote the *hilbertian sum* of the $\mathscr{X}_n$'s (i.e., $\mathscr{Y}$ is the closure of the algebraic direct sum with respect to the norm $\|\{x_n\}\|=\left(\sum_1^\infty \|x_n\|_n^2\right)^{1/2}$).

Lemma 1. *Let $\mathscr{Y}$ be the hilbertian sum of the family $\{\mathscr{X}_n\}$ of Banach spaces and let $\{T_n\}$ $(T_n\in\mathscr{L}(\mathscr{X}_n))$ be a uniformly bounded family of operators. Let $T=\bigoplus_1^\infty T_n$ be the operator defined in the usual fashion on $\mathscr{Y}$ and assume that $\|(\lambda-T_n)^{-1}\|\leqq\Phi[d_n(\lambda)-\varepsilon_n]$ for $d_n(\lambda)>\varepsilon_n$, where $\Phi(t)$ is a non-increasing function of t $(0<t<\infty)$ independent of n, $\{\varepsilon_n\}$ is a sequence of non-negative reals converging to* 0 *and $d_n(\lambda)=\operatorname{dist}[\lambda,\sigma(T_n)]$. Then $\sigma(T)=\sigma^-$, where $\sigma=\bigcup_1^\infty\sigma(T_n)$.*

Proof. Clearly, $\lambda-T$ is invertible in $\mathscr{L}(\mathscr{Y})$ if and only if $\lambda-T_n$ is invertible in $\mathscr{L}(\mathscr{X}_n)$ for every n and $\|(\lambda-T_n)^{-1}\|\leqq C$ for a constant C depending only on λ.

From our hypothesis about the growth of $\|(\lambda-T_n)^{-1}\|$, we can easily see that, given $\varepsilon>0$, $\|(\lambda-T_n)^{-1}\|\leqq\Phi[\operatorname{dist}\{\lambda,\sigma\}-\varepsilon]$ for every λ such that $\operatorname{dist}[\lambda,\sigma]>\varepsilon$ and for all $n>n_0(\varepsilon)$, whence the result follows. □

Example 1. Clearly, the function Φ must satisfy $\Phi(t) \geqq 1/t$, but the condition of Lemma 1 cannot be replaced by $\|(\lambda - T_n)^{-1}\| = O[1/d_n(t)]$. Indeed, if $H = \sum_1^\infty (1/n) P_n$, where $\{P_n\}$ is a sequence of pairwise orthogonal projections of infinite rank in the Hilbert space $\mathfrak{H}$ whose partial sums strongly converge to the identity I, then H is an hermitian operator unitarily equivalent to $H^{(\infty)}$ (the orthogonal direct sum of denumerable many copies of H), $\sigma(H) = \{0\} \bigcup_1^\infty \{1/n\} = E(H)$ ($E(\cdot)$ denotes the *essential spectrum*) and $\|W(\lambda - H)^{-1} W^{-1}\| \leqq \|W\| \, \|W^{-1}\| / d(\lambda)$ for every invertible $W \in \mathscr{L}(\mathfrak{H})$ and for every $\lambda \notin \sigma(H)$.

$\mathscr{S}(H)^-$ contains a BQT operator A such that $\sigma(A) = E(A) = \sigma(H) \cup K$, where K is an arbitrary compact connected set containing the origin ([34]), i.e., there exists a sequence $A_n = W_n H W_n^{-1}$ converging to A in the norm. It readily follows that $B = \bigoplus_1^\infty A_n$ is q.s. to H and it can be shown as in [13] that $\sigma(B) = E(B) = \sigma(A)$.

Example 2. If $\lim_{t \to 0} t^r \Phi(t) = \infty$ for every $r > 0$, then there exists a *universal quasinilpotent operator* Q in $\mathscr{L}(\mathfrak{H})$ (i.e., $\mathscr{S}(Q)^-$ contains every nilpotent) such that $\|(\lambda - Q)^{-1}\| \leqq \max\{\Phi(|t|), (1+\varepsilon)|\lambda|^{-1}\}$ (for an arbitrary prescribed $\varepsilon > 0$), $Q \cong Q^{(\infty)}$, Q is the orthogonal direct sum of denumerable many nilpotent operators acting on finite dimensional Hilbert spaces and is q.s. to a compact quasinilpotent operator (see [3; 8; 31]).

Proceeding as in Example 1 it is not difficult to construct a BQT operator $B = \bigoplus_1^\infty B_n$ q.s. to Q such that $\sigma(B) = E(B)$ is an arbitrary connected compact set containing the origin.

Theorem 1. *Assume that $T \in \mathscr{L}(\mathscr{X})$ admits a denumerable basic system of invariant subspaces $\{\mathscr{X}_n\}$ and let $T_n = T|\mathscr{X}_n$ for $n = 1, 2, \ldots$; let $\mathscr{Y}$ be the hilbertian sum of the $\mathscr{X}_n$'s and let $B \in \mathscr{L}(\mathscr{Y})$ be defined by $B = \bigoplus_1^\infty T_n$. Then B is q.s. to T, $\sigma^- \subset \sigma(B) \subset \sigma(T)$, every component of $\sigma(B)$ or $\sigma(T)$ intersects σ^-, $\sigma_p(T) = \sigma_p(B) = \bigcup_1^\infty \sigma_p(T_n) \subset \sigma$* ($\sigma_p(\cdot)$ denotes the point spectrum) *and $\sigma_p(T^*) = \sigma_p(B^*) = \bigcup_1^\infty \sigma_p(T_n^*) \subset \sigma$. Assume, moreover, that $\mathscr{X}_n$ is actually* (isomorphic with) *a Hilbert space for every n; then there exist operators L_n similar to T_n, $n = 1, 2, \ldots$, such that $A = \bigoplus_1^\infty L_n$ is q.s. to T and $\sigma(A) = \sigma^-$.*

Note. In the case when $\mathscr{X}$ is a Hilbert space and T^* is defined via inner product, $\sigma(T^*) = \sigma(T)^*$, where $K^* = \{\bar{\lambda} : \lambda \in K\}$ is the symmetric of the set $K \subset \mathbf{C}$ with respect to the real axis. In this case the corresponding inclusion should be read $\sigma_p(T^*)\subset\sigma^*$. It is convenient to remark that $\mathscr{X}_n$ can be isomorphic to a Hilbert space for every n even if $\mathscr{X}$ is not; namely, it T is a diagonal operator with respect to a Schauder basis of $\mathscr{X}$ and the $\mathscr{X}_n$'s are the one-dimensional subspaces spanned by the elements of that basis.

Proof. That B and T (and A when $\mathscr{X}_n$ is a Hilbert space for every n) are actually q.s. follows by standard arguments (see, e.g., [2; 39]). It is clear that $\lambda\in\sigma(B)$ if and only if either $\lambda\in\sigma(T_n)$ for some n or the family $\{(\lambda-T_n)^{-1}\}$ is not uniformly bounded. Now, if $\|(\lambda-T_{n(k)})x_{n(k)}\|\to 0$ $(k\to\infty)$ for a suitable subsequence $\{n(k)\}_1^\infty$ of natural numbers and for suitably chosen unitary vectors $x_{n(k)}\in\mathscr{X}_{n(k)}$, then $\lim_{k\to\infty}\|(\lambda-T)x_{n(k)}\|=0$ and therefore $\lambda\in\sigma(T)$. Hence, $\sigma^-\subset\sigma(B)$ and $\sigma(B)\diagdown\sigma\subset\sigma(T)$.

Now assume that $\mathscr{X}_n$ is a Hilbert space for every n. According to [30], for each $n=1, 2, \ldots$, there exists an operator $L_n\in\mathscr{L}(\mathscr{X}_n)$ similar to T_n such that $\|(\lambda-L_n)^{-1}\|\leqq 1/[d_n(\lambda)-1/n]$ for all λ such that $d_n(\lambda)>1/n$. Define $A\in\mathscr{L}(\mathscr{Y})$ q.s. to T and B by $A=\bigoplus_1^\infty L_n$. By Lemma 1, $\sigma(A)=\sigma^-$.

The remaining spectral inclusions follow from [13; 25; 32]. □

By using [12, Theorem 1.4], we obtain

Corollary 1. *Let T be as in Theorem 1. If $\sigma(T_n)\cap\sigma(T_m)=\emptyset$ for a pair of indices n, m then T has a nontrivial hyperinvariant subspace.*

Example 3. (The main example) Combining the arguments of the previous examples and the results of [2; 13; 29; 31; 34; 35; 39] it is possible to show that if T is a Hilbert space operator such that Lat T contains a denumerable basic system of subspaces $\{\mathfrak{R}_n\}$ such that $T_n=T|\mathfrak{R}_n$ either satisfies $A_n\oplus(\lambda+Q_n)\in\mathscr{S}(T_n)^-$ for some A_n and some nilpotent Q_n with $Q_n^n\neq 0$ or a universal quasinilpotent, or $\sigma(T_n)$ contains more than n points, then given an arbitrary compact set $K\subset\mathbf{C}$ such that every $\lambda\in K\diagdown\sigma^-$ belongs to a component of K that intersects $\sigma_\infty=\bigcap_{m=1}^\infty[\bigcup_{n=m}^\infty\sigma(T_n)]^-$, then there exist A and B q.s. to T such that $\sigma(A)=K\cup\sigma^-$ and $\sigma(B)=K\cup\sigma(T)$. The details of the construction are left to the reader.

Remarks. a) Let $\mathscr{B}$ be a Banach algebra with identity. It is well known that the mapping $a\to\sigma(a)$ from $\mathscr{B}$ into the family of nonempty compact subsets of $\mathbf{C}$ is upper semi-continuous with respect to the Hausdorff metric, but it is not continuous, in general ([5; 25; 29; 40; 42; 44]). In certain special cases (e.g., $a=\lim a_n$ for a *commutative* sequence $\{a_n\}$, or $\sigma(a)=a$ totally disconnected set, etc.) it is possible to prove that $a\to\sigma(a)$ is actually a continuous mapping. By a minor modification of the proof of Lemma 1, we can obtain the following sufficient condition: "If $a=\lim a_n$ for a sequence $\{a_n\}$ satisfying the conditions of Lemma 1, then $\sigma(a)=\lim\sigma(a_n)$ (in the Hausdorff metric)". Examples 1 and 2 show that this condition cannot be too relaxed.

b) In Lemma 1 and Theorem 1: The results remain true if the hilbertian sum is replaced by $\|\{x_n\}\|=(\sum_{n=1}^{\infty}\|x_n\|^p)^{1/p}$ for some p, $1\leqq p<\infty$, etc.

c) If T is decomposable, then $\sigma(T)\subset\sigma(A)$ for every A q.s. to T ([10]). Furthermore, if $\mathfrak{M}\in\mathrm{Lat}\,T$ and A is q.s. to T, then $\sigma(T|\mathfrak{M})\cap\sigma(A)\neq\emptyset$ ([14]); thus, if for every $\lambda\in\sigma(T)$ and every $\varepsilon>0$ there exists an $\mathfrak{M}_{\lambda,\varepsilon}\in\mathrm{Lat}\,T$ such that $\sigma(T|\mathfrak{M}_{\lambda,\varepsilon})\subset\subset\Delta(\lambda,\varepsilon)=\{z\colon|\lambda-z|<\varepsilon\}$, then it readily follows that $\sigma(T)\subset\sigma(A)$ for every A q.s. to T. The hyponormal operators have the same property ([9]). The spectral inclusion could be strict, e.g., for the operators of Examples 1,2,3. However, by combining the results of [23] and the examples of [28] it is possible to show that for every infinite dimensional separable Banach space $\mathscr{X}$, there exist operators $A, Q\in\mathscr{L}(\mathscr{X})$ such that A and Q are nuclear operators, Q is quasinilpotent, $\sigma(A)$ is the union of $\{0\}$ and a sequence of points converging "very fast" to 0, $\mathscr{A}(A)$ and $\mathscr{A}(Q)$ are strictly cyclic algebras, $\mathscr{A}(A)$ ($\mathscr{A}(Q)$, resp.) is semisimple (a radical algebra, resp.; see definitions in [42]), every L q.s. to A (to Q, resp.) is actually similar to it and it has the same spectrum as A (as Q, resp.). Moreover, for every finite m, $\mathrm{Lat}\,A$ contains a basic system $\{\mathscr{X}_n\}_1^m$ of invariant subspaces, which are maximal spectral subspaces for the decomposable operator A ([10]); however, $\mathrm{Lat}\,A$ does not contain any denumerable basic system of subspaces (see [28]).

d) Every subspace in a basic system of invariant subspaces of $T\in\mathscr{L}(\mathscr{X})$ is actually *bi-invariant*. Many examples regarding operators T such that $\sigma(A)\neq\sigma(T)$ for some A q.s. to T deal with operators having a denumerable basic system of *hyperinvariant* subspaces. This is not always the case: indeed, a straightforward computation shows that for the q.s. operators A and T involved in the example of Hoover [39], every pair of non-trivial hyperinvariant subspaces of T (or A) has a non-trivial intersection.

e) There is little hope to improve [12, Theorem 1.4] or Corollary 1. Indeed, if U denotes the bilateral shift "multiplication by $e^{i\theta}$" in L^2 (Unit circle, Lebesgue measure) and $u(e^{i\theta})=\mathrm{sign}\,\theta$ $(-\pi<\theta<+\pi)$, then H^2 and uH^2 are invariant (but not bi-invariant!) subspaces of U such that $H^2\cap uH^2=\{0\}$, $L^2=H^2\vee uH^2$, but (by Apostol's result; [2]) U cannot be q.s. to $(U|H^2)\oplus(U|uH^2)$.

f) [15, Theorem 2.1] admits the following mild generalization, which follows from Theorem 1 and the same proof as in [15]: If $T\in\mathscr{L}(\mathfrak{K})$ and $\mathrm{Lat}\,T$ contains a basic system of subspaces $\{\mathfrak{K}_n\}$ such that $T_n=T|\mathfrak{K}_n$ is a spectral operator for every n, then T is q.s. to a spectral operator.

3. **The subsets** (A), (B) **and** (C) **are dense in** $\mathscr{L}(\mathfrak{K})$. From this point on, we shall only consider Hilbert space operators. The density of (A) follows from [19].

Lemma 2. *Given* $T\in\mathscr{L}(\mathfrak{K})$ *and* $\varepsilon>0$, *there exists* $T_\varepsilon\in\mathscr{L}(\mathfrak{K})$ *such that* $\|T-T_\varepsilon\|<\varepsilon$ *and* T_ε *is similar to* $(\lambda+Q)\oplus C$, *where* $\sigma(\lambda+Q)$ *lies in the unbounded component of* $\mathbf{C}\setminus\sigma(C)$, $E(T)=E(C)$ *and* Q *is an arbitrary operator such that* $\sigma(Q)\subset\Delta(0,\varepsilon/5)$.

Proof. Proceeding as in [45], we can find an $L\in\mathscr{L}(\mathfrak{K})$ such that $\|T-L\|<3\varepsilon/4$ and

$$L=\begin{pmatrix}\lambda I & B\\ 0 & C\end{pmatrix}$$

with respect to an orthogonal direct sum decomposition $\mathfrak{K}=\mathfrak{K}_\lambda\oplus\mathfrak{K}_\lambda^\perp$ of $\mathfrak{K}$ into two infinite dimensional subspaces, where $\operatorname{dist}[\lambda,\sigma(T)]=\operatorname{dist}[\lambda,\sigma(C)]=\varepsilon/2$ and λ lies in the unbounded component of $\mathbf{C}\setminus\sigma(C)$.

By the corollary of Rota [43] (see also [30]), we can find a Q' similar to Q such that $\|Q'\|<\varepsilon/4$. Then

$$T_\varepsilon=\begin{pmatrix}\lambda+Q' & B\\ 0 & C\end{pmatrix}$$

is similar to $(\lambda+Q)\oplus C$, by Rosenblum's corollary ([41, Corollary 0.15]) and $\|T-T_\varepsilon\|\leqq\|T-L\|+\|Q'\|<\varepsilon$. □

As in Hoover [39], we can find two q.s. operators Q_1 and Q_2 such that Q_1 is quasinilpotent and $\sigma(Q_2)=\Delta(0,\varepsilon/6)^-$, and $\|Q_j\|<\varepsilon/4$, $j=1,2$. By using the results of [19], C can be replaced by an operator $C_\varepsilon\in(BQT)_{qs}$ with the same spectrum as C such that $\|C-C_\varepsilon\|<\varepsilon$. Then the operator $T_{\varepsilon j}$ given by

$$T_{\varepsilon j}=\begin{pmatrix}\lambda+Q_j & B\\ 0 & C_\varepsilon\end{pmatrix}$$

satisfies $\|T-T_{\varepsilon j}\|<\varepsilon$, $j=1,2$, and it is immediate from our construction that $T_{\varepsilon 1}$ and $T_{\varepsilon 2}$ are q.s. operators of the class $(BQT)_{qs}$.

Since $\sigma(T_{\varepsilon 1})$ is a proper subset of $\sigma(T_{\varepsilon 2})$, it follows at once that (B) and (C) are dense in $\mathscr{L}(\mathfrak{K})$.

Given $T\in\mathscr{L}(\mathfrak{K})$, let T_ε be constructed as in Lemma 2 with $Q=V=$the Volterra operator, and let W be an invertible operator such that $T_\varepsilon=W[(\lambda+V)\oplus\oplus C]W^{-1}$. Then $\mathscr{A}(T_\varepsilon)=W[\mathscr{A}(V)\oplus\mathscr{A}(C)]W^{-1}$, $\operatorname{Alg\,Lat}\mathscr{A}(T_\varepsilon)=W[\operatorname{Alg\,Lat}\mathscr{A}(V)\oplus\oplus\operatorname{Alg\,Lat}\mathscr{A}(C)]W^{-1}$ and similarly for the other three algebras naturally associated with T_ε (all these facts can be easily checked by using the results of [41]); moreover, $\operatorname{appr}(T_\varepsilon)''=W[\operatorname{appr}(V)''\oplus\operatorname{appr}(C)'']W^{-1}$ ([24]).

Since $\mathscr{A}(V)=\mathscr{A}^a(V)=\mathscr{A}''(V)=\mathscr{A}'(V)\neq\operatorname{Alg\,Lat}V$ ([41]) and $\mathscr{A}''(V)\neq\neq\operatorname{appr}(V)''$ (see [41] or [20, Proposition 6]), it follows that none of the four algebras associated with T_ε is reflexive and $\operatorname{appr}(T_\varepsilon)''\neq\mathscr{A}''(T_\varepsilon)$. Thus, we have

Corollary 2. *The complements of the sets* (G), (H), (I), (J) *and* (K) *are dense in* $\mathscr{L}(\mathfrak{K})$.

4. (D) **is dense in** $\mathscr{L}(\mathfrak{R})$. The main ingredient is the construction of a large family of operators with a strictly cyclic double commutant.

Let Ω be a nonempty bounded connected open subset of the plane such that $\partial\Omega$ (the boundary of Ω) consists of finitely many pairwise disjoint regular analytic Jordan curves (We shall say that "Ω is an open set with analytic boundary" or "$\partial\Omega$ is analytic" as a shorthand notation) and let $\Lambda=\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ be a finite subset of $\mathbf{C}\setminus\Omega^-$ having exactly one point in every component of this last set. Let $\varepsilon>0$ be small enough so that $\Lambda\cap(\Omega^-+\varepsilon t)=\emptyset$ (where $K+\lambda=\{z+\lambda\colon z\in K\}$, $K\subset\mathbf{C}$) for every $t\in[0,1]$ and define $\Gamma=\{(z,t)\in\mathbf{C}\times(0,1)\colon z-\varepsilon t\in\partial\Omega\}$. It is apparent that there exists an analytic diffeomorphism $\varphi\colon\{(z,t)\in\mathbf{C}\times(-1,2)\colon z-\varepsilon t\in\partial\Omega\}\to\Omega_1$, where Ω_1 is the union of m open annulus with pairwise disjoint closures in the plane; then $\varphi|\Gamma\colon\Gamma\to\Omega_0\overset{\mathrm{def}}{=}\varphi(\Gamma)$ is an analytic diffeomorphism such that if dm_0 denotes the planar Lebesgue measure on Ω_0 and dm_Γ is the area measure on Γ induced by Lebesgue measure in $\mathbf{R}^3$, then there exists $\delta, 0<\delta<1$, such that $\delta m_0[\varphi(B)]\leqq m_\Gamma(B)\leqq(1/\delta)m_0[\varphi(B)]$ for every Borel set $B\subset\Gamma$; moreover, φ can be chosen to be a conformal mapping.

The Sobolev space $\mathbf{W}^{2,2}(\Omega_0)$ of all distributions u on Ω_0 whose distributional partial derivatives of order $m, 0\leqq m\leqq 2$, belong to $L^2(\Omega_0, dm_0)$ can be identified with a Banach algebra (under an equivalent norm) of continuous functions on Ω_0^- (see [1, Chapter V]) and it is clear that φ induces an isomorphism between this space and $\mathbf{W}_\infty=\mathbf{W}^{2,2}(\Gamma)$ (defined in the obvious way on the analytic differentiable manifold Γ). Furthermore, by using this isomorphism, it is easily seen that there exists a constant C such that, given $f, g\in\mathbf{W}_\infty$, the pointwise product $(fg)(z,t)=f(z,t)\cdot g(z,t)$ defines an element of $\mathbf{W}_\infty$ and $\|fg\|\leqq C\|f\|\,\|g\|$ (where $\|\cdot\|$ denotes the norm in $\mathbf{W}_\infty$); hence, $\mathbf{W}_\infty$ is a semisimple Banach algebra with identity $e(z,t)\equiv 1$, under an equivalent norm. The Gelfand spectrum $\mathscr{M}(\mathbf{W}_\infty)$ can be naturally identified (via point evaluations; see [1; 21]) with Γ^-.

Let $\mathbf{A}_\infty=\mathbf{A}^{2,2}(\Gamma)$ be the closure in $\mathbf{W}_\infty$ of the functions of the form

$$f(z,t)=\sum_{k=0}^{n} t^k f_k(z), \quad n=1,2,\dots, \tag{\#}$$

where the f_k's are rational functions with poles in a subset of Λ (these are the "analytic elements" of $\mathbf{W}_\infty$). By using the maximum modulus principle and Runge's theorem (see, e.g., [21]), it is easily seen that every $f\in\mathbf{A}_\infty$ can be continuously extended to a unique function defined on $\Xi=\{(z,t)\in\mathbf{C}\times[0,1]\colon z-\varepsilon t\in\Omega^-\}$, analytic with respect to $z\in\Omega+\varepsilon t$ for every $t\in[0,1]$ and, on the other hand, every function $f(z,t)$ satisfying these conditions such that $f|\Gamma\in\mathbf{W}_\infty$, is an element of $\mathbf{A}_\infty$.

$\mathbf{A}_\infty$ is a subspace of $\mathbf{W}_\infty$ invariant under $T=M_z\in\mathscr{L}(\mathbf{W}_\infty)$ defined by $Tf(z,t)=zf(z,t)$ (here and in what follows, M_g denotes the operator "multiplication by g"). Moreover, $\mathbf{A}_\infty$ is a Banach algebra with identity e, and $\mathcal{M}(\mathbf{A}_\infty)$ can be naturally identified with Ξ.

Let $L=T|\mathbf{A}_\infty$ and let Pr: $\mathbf{C}\times\mathbf{R}\to\mathbf{C}$ be the projection onto $\mathbf{C}$ $(Pr(z,t)=z)$; then

Lemma 3. *With the above notation:*

(i) $\sigma(T)=E_l(T)=E_r(T)=E_l(L)=Pr(\Gamma^-)$, *where* $E_l(\cdot)$ $(E_r(\cdot)$, resp.) *denotes the left (right,* resp.*) essential spectrum.*

(ii) $\sigma(L)=E_r(L)=Pr(\Xi)$.

(iii) $\operatorname{Ker}(\lambda-L)=\{0\}$ *and* $\dim\operatorname{Ker}(\lambda-L)^*=\infty$ *(so that* $\operatorname{ind}(\lambda-L)=-\infty$*) for every* $\lambda\in\sigma(L)\setminus E_l(L)$.

(iv) $\mathscr{A}''(L)=\mathscr{A}'(L)=\{M_g\colon g\in\mathbf{A}_\infty\}$, *i.e., the double commutant of L is the maximal abelian subalgebra of $\mathscr{L}(\mathbf{A}_\infty)$ consisting of all multiplications by elements of $\mathbf{A}_\infty$ and this is a strictly cyclic algebra with strictly cyclic vector e.*

Proof. (i), (ii) and (iii) follow from the previous observations. The proof is left to the reader.

(iv) By using several well known results about strictly cyclic algebras ([26; 27; 37]), it suffices to show that, if $A\in\mathscr{A}'(L)$, then $A=M_g$, where $g=Ae$. The remaining of the proof is an "ad hoc" modification of an argument used in [28].

Given $\eta,\tau\in[0,1]$, $\eta\neq\tau$, choose δ, $0<\delta<|\eta-\tau|/8$, and let $h_\eta(z,t)\in\mathbf{A}_\infty$ be the restriction to Γ^- of the function defined by

$$h_\eta(z,t)=\begin{cases}0 & \text{outside } (\eta-3\delta,\eta+3\delta),\\ (t-\eta+3\delta)^2/2\delta^2 & \text{in } [\eta-3\delta,\eta-2\delta],\\ 1-(t-\eta+\delta)^2/2\delta^2 & \text{in } [\eta-2\delta,\eta-\delta],\\ 1 & \text{in } [\eta-\delta,\eta+\delta],\\ 1-(t-\eta-\delta)^2/2\delta^2 & \text{in } [\eta+\delta,\eta+2\delta],\\ (t-\eta-3\delta)^2/2\delta^2 & \text{in } [\eta+2\delta,\eta+3\delta].\end{cases}$$

Define $h_\tau(z,t)=h_\eta(z,t-\eta+\tau)$ and let $\psi\colon[0,1]\to[\tau-4\delta,\tau+4\delta]\cap[0,1]$ be an arbitrary C^∞ bijection such that $\psi(t)=t$ in $[\tau-3\delta,\tau+3\delta]\cap[0,1]$ and $\min\{\psi'(t)\colon t\in[0,1]\}>0$.

Define $\mathbf{W}_{\tau,\delta}=\mathbf{W}^{2,2}(\{(z,t)\in\Gamma\colon|t-\tau|<4\delta\})$ exactly in the same way as $\mathbf{W}_\infty$ and let $T_{\tau,\delta}$ be the "multiplication by z" in this new space. Let $\mathbf{A}_{\tau,\delta}$ be the subalgebra of the "analytic elements" of $\mathbf{W}_{\tau,\delta}$ (defined in the obvious way) and $L_{\tau,\delta}=T_{\tau,\delta}|\mathbf{A}_{\tau,\delta}$.

The properties of ψ make it clear that $S\colon\mathbf{A}_{\tau,\delta}\to\mathbf{A}_\infty$ defined by $Sf(z,t)=$ $=f(z+\varepsilon[\psi(t)-t],\psi(t))$ is a (not necessarily isometric) isomorphism of Hilbert spaces.

Our choice of δ makes it possible to find a disc $\Delta=\Delta(\lambda(\eta,\tau),\varepsilon\delta/2)$ contained in $\Omega+\varepsilon\eta$ such that $\Delta^-\cap\sigma(L_{\tau,\delta})=\emptyset$.

Finally, let R: $\mathbf{A}_\infty \to H^2(\Delta)$ be the "restriction in the η-fiber" mapping defined by $Rf(z)=f(z, \eta)|_{z\in\Delta}$ and let L_Δ be the "multiplication by z" in $H^2(\Delta)$.

Clearly, if M_η and M_τ are the multiplications by h_η and h_τ, respectively, then $M_\eta A M_\tau \in \mathcal{A}'(L)$, so that $L(M_\eta A M_\tau)-(M_\eta A M_\tau)L=0$ whence we obtain $0=RLM_\eta AM_\tau S-RM_\eta AM_\tau LS=L_\Delta(RM_\eta AM_\tau S)-(RM_\eta AM_\tau S)L_{\tau,\delta}$ (Beware! $LS\neq \neq SL_{\tau,\delta}$; however, it is not difficult to check that $\psi(t)=t$ in $[\tau-3\delta, \tau+3\delta]\cap[0, 1]$ yields $M_\tau LS=M_\tau SL_{\tau,\delta}$).

Since $\sigma(L_\Delta)=\Delta^-$ is disjoint from $\sigma(L_{\tau,\delta})$ by construction, it follows from Rosenblum's corollary ([41, Corollary 0.13]) that $RM_\eta AM_\tau S=0$; moreover, since S is an isomorphism, $RM_\eta AM_\tau=0$. Since Ω is connected, the vanishing of $f(z, \eta)$ on Δ implies that $f(z, \eta)\equiv 0$, whence we conclude that the value of $Af(z, \tau)$ only depends on the values of $f(z, t)$ for t in a neighborhood of τ.

We shall need a little more: A straightforward computation shows that $\|(t-\tau)^k h_\tau(z, t)\|\to 0$ as $\delta\to 0$, uniformly with respect to k ($k=1, 2, \ldots$). Let f be any function of the form (#) and let $F(z, t)=f(z, \tau)$; then

$$f(z, t) = F(z, t)+\sum_{k=1}^{n}(t-\tau)^k f_k(z),$$

where the f_k's are rational functions of z with poles in a subset of Λ. Since A commutes with M_z, it is clear that $AM_F=M_FA$ and $AM_{f_k}=M_{f_k}A$ for $k=1, 2, \ldots, n$, and therefore $AF(z, t)=AM_Fe(z, t)=[M_F(Ae)](z, t)=F(z, t)g(z, t)=g(z, t)f(z, \tau)$, which is equal to $g(z, \tau)f(z, \tau)$ for $t=\tau$. Hence,

$$Af(z, \tau) = AF(z, \tau)+\sum_{k=1}^{n} f_k(z)\lim_{\delta\to 0} A[(t-\tau)^k h_\tau](z, \tau) =$$

$$= g(z, \tau)f(z, \tau), \quad \text{for every} \quad \tau\in[0, 1].$$

Therefore, $Af(z, t)=g(z, t)f(z, t)$ on Γ^- for every f of the form (#). By continuity, we conclude that $A=M_g$. □

By a formal repetition of the proof of [28, Theorem 8] and the above result, we can easily obtain

Lemma 4. *Let Ω, ε and A be as in Lemma 3 and let n be a positive integer. Define $\mathbf{W}_n=\bigoplus_{k=1}^{n}\mathbf{W}^{2,1}(\partial\Omega+k\varepsilon/n, dm_k)$, where dm_k is the "arc length measure" on $\partial\Omega+k\varepsilon/n$ and $\mathbf{W}^{2,1}(\partial\Omega+k\varepsilon/n, dm_k)$ is the Sobolev space of all distributions u on $\partial\Omega+k\varepsilon/n$ with distributional derivative (with respect to "arc length") in $L^2(\partial\Omega+k\varepsilon/n, dm_k)$, with the norm*

$$\|f\| = \Big\{\int_{\partial\Omega+k\varepsilon/n}[|f(z)|^2+|df/dm_k(z)|^2]\,dm_k\Big\}^{1/2},$$

and let $\mathbf{A}_n$ *be the subspace of "analytic elements" of* $\mathbf{W}_n$ (i.e., $\mathbf{A}_n = \mathbf{W}_n$*-closure* $\{(f_1, f_2, \dots, f_n)\colon f_k$ is rational with poles in a subset of $\Lambda\}$).

Then $\mathbf{W}_n$ *and* $\mathbf{A}_n$ *are semisimple Banach algebras of continuous functions with identity (under an equivalent norm),* $\mathcal{M}(\mathbf{W}_n)$ ($\mathcal{M}(\mathbf{A}_n)$, resp.) *can be naturally identified with* $\bigcup_{k=1}^{n} (\partial\Omega + k\varepsilon/n) \times \{k/n\}$ $\left(\bigcup_{k=1}^{n} (\Omega^- + k\varepsilon/n) \times \{k/n\}\right.$, resp.$\left.\right) \subset \mathbf{C} \times [0, 1]$.

Furthermore, if $T_n = M_z$ *in* $\mathbf{W}_n$*, then* $\mathbf{A}_n$ *is invariant under* T_n*, and* T_n *and its restriction* $L_n = T_n | \mathbf{A}_n$ *satisfy*

(i) $\sigma(T_n) = E_l(T_n) = E_r(T_n) = E_l(L_n) = E_r(L_n) = Pr[\mathcal{M}(W_n)]$.

(ii) $\sigma(L_n) = Pr[\mathcal{M}(\mathbf{A}_n)]$.

(iii) $\operatorname{Ker}(\lambda - L_n) = \{0\}$ *and* $\dim \operatorname{Ker}(\lambda - L_n)^* = n$ (*so that* $\operatorname{ind}(\lambda - L_n) = -n$)

for every $\lambda \in \bigcap_{k=1}^{n} (\Omega + k\varepsilon/n) \subset \sigma(L_n) \setminus E(L_n)$.

(iv) $\mathcal{A}'(L_n) = \mathcal{A}''(L_n) = \{M_g\colon g \in \mathbf{A}_n\}$ *is a maximal abelian strictly cyclic subalgebra of* $\mathcal{L}(\mathbf{A}_n)$.

The proof is left to the reader.

Given Ω with analytic boundary, $\varepsilon > 0$ and Λ as indicated, and an index n, $-\infty \leqq n < 0$, we shall denote by $T(\Omega, \varepsilon, n)$ and $L(\Omega, \varepsilon, n)$ the operators defined by Lemma 3 (for $n = -\infty$) or by Lemma 4 (for $-\infty < n < 0$). If $0 < n \leqq +\infty$, we shall use the adjoint operators $T(\Omega^*, \varepsilon, -n)^*$ and $L(\Omega^*, \varepsilon, -n)^*$.

Now we are in a position to prove the main result of this paper.

Theorem 3. *The subset (D) of those operators T similar to* $A \oplus B$*, where*

(i) $\sigma(A) \cap \sigma(B) = \emptyset$;

(ii) $\mathcal{A}''(A)$ *and* $\mathcal{A}''(B^*)$ *are strictly cyclic algebras;*

(iii) $\lambda_A - A$ *and* $\lambda_B - B^*$ *are semi-Fredholm operators of index* $-\infty$, *for suitably chosen* λ_A *and* λ_B;

(iv) $\mathcal{S}(A \oplus B) = \mathcal{S}_{qs}(A \oplus B)$ *and this set does not intersect* $(BQT)_{qs}$*; is dense in* $\mathcal{L}(\mathfrak{R})$.

Proof. The result follows by modifying the proofs in [6].

By [6, Proposition 1.4] (Indeed, by a minor modification of it), given $T \in \mathcal{L}(\mathfrak{R})$ and $\varepsilon > 0$, there exists an operator T_1 such that $\|T - T_1\| < \varepsilon$ and

$$T_1 = \begin{pmatrix} N_1 & 0 & * & * & * \\ 0 & N_2 & * & * & * \\ 0 & 0 & S_1 & * & * \\ 0 & 0 & 0 & N_3 & 0 \\ 0 & 0 & 0 & 0 & N_4 \end{pmatrix}$$

(with respect to a suitable orthogonal direct sum decomposition of $\mathfrak{K}$ into five subspaces), where

a) N_j is normal and $\sigma(N_j)=E(N_j)$ for $j=1, 2, 3, 4$;

b) $\sigma(N_2)\cup\sigma(N_3)$ is the closure of a nonempty open subset Ω_0 with analytic boundary;

c) $\sigma(N_1)\cap\sigma(N_4)=\emptyset$, $\sigma(N_1)$ and $\sigma(N_4)$ are disjoint unions of pairwise disjoint regular analytic Jordan curves, $\sigma(N_1)\subset\partial\sigma(N_2)\cap\partial\Omega_0$ and $\sigma(N_4)\subset\partial\sigma(N_3)\cap\partial\Omega_0$; $\sigma(N_1)$ ($\sigma(N_4)$, resp.) is contained in the open set $\{\lambda: (\lambda-T)$ is semi-Fredholm of negative (positive, resp.) index$\}$;

d) S_1 is similar to a direct sum $F\oplus S_2$, where F is a normal operator with simple eigenvalues (i.e., cyclic) acting on a finite dimensional subspace, such that $\sigma(F)\cap[\sigma(N_2)\cup\sigma(N_3)\cup\sigma(S_2)]=\emptyset$, and $\partial\sigma(S_2)\subset\Omega_0^-$;

e) The Weyl spectrum $w(T)$ of T satisfies the inclusions $w(T)\overset{\text{def}}{=}$ $=\sigma(T)\setminus\{\lambda: (\lambda-T)$ is a Fredholm operator of index $0\}\subset\sigma(N_2)\cup\sigma(N_3)\cup\sigma(S_2)\subset$ $\subset w(T)_\varepsilon$, where $K_\varepsilon=\{\lambda: \operatorname{dist}(\lambda, K)\leqq\varepsilon\}$ $(K\subset\mathbf{C})$;

f) $\text{min. ind}\,(\lambda-S_2)\overset{\text{def}}{=}\min\{\dim\operatorname{Ker}(\lambda-S_2), \dim\operatorname{Ker}(\lambda-S_2)^*\}=0$ for every λ such that $(\lambda-S_2)$ is semi-Fredholm.

Clearly, $\sigma(T_1)$ is the disjoint union of its clopen subsets $\sigma(F)$ and $\sigma(T_1)\setminus\sigma(F)$ so that, by ROSENBLUM [41, Corollary 0.15), T_1 is similar to $F\oplus T_2$, where

$$T_2=\begin{pmatrix} N_1 & * & * & * & * \\ 0 & N_2 & * & * & * \\ 0 & 0 & S_2 & * & * \\ 0 & 0 & 0 & N_3 & * \\ 0 & 0 & 0 & 0 & N_4 \end{pmatrix}.$$

According to c), $N_1=\bigoplus_{k=1}^{m} N_{1k}$ $\left(N_4=\bigoplus_{j=1}^{p} N_{4j}\right)$, where $\sigma(N_{1k})=E(N_{1k})$ $(\sigma(N_{4j})=$ $=E(N_{4j})$, resp.) is the boundary of a unique component Ω_k (Ω_j, resp.) of the semi-Fredholm domain of T_2, where $\operatorname{ind}(\lambda-T_2)=n_k<0$ $(=n_j>0$, resp.) for all $\lambda\in\Omega_k$, $k=1, 2, \ldots, m$ $(\lambda\in\Omega_j, j=1, 2, \ldots, p$, resp.).

Let $\Lambda M=\{\lambda_1, \lambda_2, \ldots, \lambda_q, \mu_1, \mu_2, \ldots, \mu_q\}$ be a finite set having exactly two points, λ_h and μ_h, in each of the q components of Ω_0. Replacing, if necessary, ε by an ε', $0<\varepsilon'<\varepsilon$, we can assume that the three sets $\sigma(F)$, $(\Lambda M)_\varepsilon$ and $(\Omega_0)_\varepsilon$ are pairwise disjoint.

Let $T(\Omega_k, \varepsilon, n_k)$ $(k=1, 2, \ldots, m)$ and $T(\Omega_j^*, \varepsilon, -n_j)^*$ $(j=1, 2, \ldots, p)$ be the operators constructed as above indicated. Since $d_H\{\sigma(N_{1k}), \sigma[T(\Omega_k, \varepsilon, n_k)]\}\leqq\varepsilon$ (d_H denotes the Hausdorff distance), it follows from [35] that there exists T_k' similar to $T(\Omega_k, \varepsilon, n_k)$ such that $\|T_k'-N_{1k}\|<2\varepsilon$, $k=1, 2, \ldots, m$. Analogously, there exists T_j'' similar to $T(\Omega_j^*, \varepsilon, -n_j)^*$ such that $\|T_j''-N_{4j}\|<2\varepsilon$, $j=1, 2, \ldots, p$; thus, if

$M_1=\bigoplus_{k=1}^{m} T'_k$ and $M_4=\bigoplus_{j=1}^{p} T''_j$, and T_3 is the operator obtained from T_2 by replacing N_1 by M_1 and N_4 by M_4, then $\|T_2-T_3\|<2\varepsilon$.

It is clear that M_1 has an invariant subspace $\mathfrak{M}_1$ such that $L_1=M_1|\mathfrak{M}_1$ is similar to $\bigoplus_1^m L(\Omega_k, \varepsilon, n_k)$ and that M_4 has an invariant subspace $\mathfrak{M}_4$ such that the *compression* L_4 of M_4 to $\mathfrak{M}_4^\perp$ is similar to $\bigoplus_1^p L(\Omega_j^*, \varepsilon, -n_j)^*$. Since the spectra of the components of these direct sums are pairwise disjoint, it follows as in the proof of Corollary 2 that $\mathscr{A}''(L_1)$ and $\mathscr{A}''(L_4^*)$ are strictly cyclic operator algebras.

Proceeding exactly as in the proof of [6, Proposition 2.1], we find out that

$$T_3 = \begin{pmatrix} L_1 & * & * \\ 0 & S_3 & * \\ 0 & 0 & L_4 \end{pmatrix}$$

(with respect to a suitable orthogonal direct sum decomposition), where $S_3\in BQT$ and $\sigma(S_3)=E(S_3)=\Omega_0^-$.

Let $C_1=\bigoplus_{h=1}^{q} (\lambda_h+L[\Delta(\lambda_h, \varepsilon/2), \varepsilon/2, -\infty])$ and $C_4=\bigoplus_{h=1}^{q} (\mu_h+L[\Delta(\mu_h, \varepsilon/2), \varepsilon/2, -\infty]^*)$. By using the results of [34; 35] and Rosenblum [41, Corollary 0.15], we can find an operator

$$S_4 = \begin{pmatrix} C'_1 & * \\ 0 & C'_4 \end{pmatrix},$$

with C'_i similar to C_i, $i=1, 4$, such that $\|S_3-S_4\|<\varepsilon$, so that if T_4 is the operator obtained from T_3 by replacing S_3 by S_4, then a formal repetition of previous arguments shows that $\|T_3-T_4\|<\varepsilon$ and T_4 is similar to $L_1\oplus C'_1\oplus C'_4\oplus L_4$ which, in turn, is similar to $A_0\oplus B$, where $A_0=\{\bigoplus_1^m L(\Omega_k, \varepsilon, n_k)\}\oplus C_1$ and

$$B = C_4\oplus\{\bigoplus_1^p L(\Omega_j^*, \varepsilon, -n_j)^*\}.$$

Thus, if $A=F\oplus A_0$, it readily follows that there exists an operator T_5 similar to $A\oplus B$ such that $\|T-T_5\|<4\varepsilon$.

Since A and B clearly satisfy (i)—(iv), we are done. □

Corollary 3. *$(E)_{mn}$, (F), (G), (H), (I) and (J) are dense in $\mathscr{L}(\mathfrak{H})$.*

Proof. The proof will be just sketched. Repeat exactly the same proof as above replacing ΛM by $\Lambda MN\Pi=\{\lambda_1, \ldots; \lambda_q, \mu_1, \ldots, \mu_q, \nu_1 \ldots, \nu_q; \pi_1, \ldots, \pi_4\}$ with the same characteristics as ΛM and four points, $\lambda_h, \mu_h, \nu_h, \pi_h$, in each component of Ω_0.

$(E)_{mn}$: Replace A and B by $A\oplus\{\bigoplus_{h=1}^{q}\nu_h I_m\}$ and $B\oplus\{\bigoplus_{h=1}^{q}\pi_h I_n\}$, resp., where I_m (I_n) is the identity on a Hilbert space of algebraic dimension m (n, resp.), and use the results of [27].

(F) Replace A by $A\oplus\{\bigoplus_{h=1}^{q}(\nu_h+Q)\}$, where Q is any nilpotent of infinite rank. The result follows as in Theorem 3 by using the results of [3].

(I) and (J): These two cases follow at once from Theorem 3, the fact that $\mathbf{A}_\infty$ and $\mathbf{A}_n$ are *semisimple* Banach algebras and [47]; it is easily seen that $\mathscr{A}''(A)=\mathscr{A}'(A)$ and $\mathscr{A}''(B)=\mathscr{A}'(B)$ are reflexive.

(G) and (H): These two cases follow at once from the above observations abou $\mathscr{A}''(A)$ and $\mathscr{A}''(B)$ and the results of [37; 38]. □

Remark. An alternative proof for the cases $(G)-(J)$ can be obtained by using the Apostol—Morrel dense class $C_0(\mathfrak{R})$ (see definition and properties in [6]) and the results of [41].

5. (K) **is dense in** $\mathscr{L}(\mathfrak{R})$. The proof is a "trivialization" of that of the case (D).

Lemma 5. *Let Ω be an open set with analytic boundary, let $\Gamma_0=\partial\Omega\times(0,1)$ and $\Xi_0=\Omega^-\times[0,1]$.* $\mathbf{W}_{0\infty}=\mathbf{W}^{2,2}(\Gamma_0)$ (defined as in Section 4) *has the same properties as* $\mathbf{W}_\infty$ *and the subalgebra* $\mathbf{A}_{0\infty}$ *of "analytic elements" of* $\mathbf{W}_{0\infty}$ ($\mathbf{A}_{0\infty}=\{f\in\mathbf{W}_{0\infty};\ f(z,t)$ is analytic with respect to $z\in\Omega$ for every $t\in[0,1]\}$) *has the same properties as* $\mathbf{A}_\infty$.

If $T_0=M_z$ *in* $\mathbf{W}_{0\infty}$ *and* $L_0=T_0|\mathbf{A}_{0\infty}$, *then:*

(i) $\sigma(T_0)=E_l(T_0)=E_r(T_0)=E_l(L_0)=\partial\Omega$.

(ii) $\sigma(L_0)=E_r(L_0)=\Omega^-$.

(iii) $\operatorname{Ker}(\lambda-L_0)=\{0\}$ *and* $\dim\operatorname{Ker}(\lambda-L_0)^*=\infty$ *(so that* $\operatorname{ind}(\lambda-L_0)=-\infty$*) for every* $\lambda\in\Omega$.

(iv) $\mathscr{A}'(L_0)\supset\{M_g\colon g\in\mathbf{A}_{0\infty}\}$, *so that* $\bar\mu[\mathscr{A}'(L_0)]=1$.

(v) $\mathscr{A}^a(L_0)=\mathscr{A}''(L_0)=\{M_g\colon g\in\mathbf{A}_{0\infty},\ g(z,t)$ is constant with respect to t for every (fixed) $z\in\Omega^-\}$ = *norm-closure of the rational functions of* L_0 *with poles outside* Ω^-.

(vi) $\operatorname{appr}(L_0)''=\mathscr{A}''(L_0)$.

Proof. The statements relative to $\mathbf{W}_{0\infty}$ and $\mathbf{A}_{0\infty}$ (in particular, $\mathscr{M}(\mathbf{W}_{0\infty})\approx\Gamma_0^-$ and $\mathscr{M}(\mathbf{A}_{0\infty})\approx\Xi_0$) can be proved exactly as in the previous section. Now (i), (ii) and (iii) are clear and (iv) is obvious.

(v) Let $A\in\mathscr{A}''(L_0)$. Since A commutes with the maximal abelian algebra of all multiplications by the elements of $\mathbf{A}_{0\infty}$, A must be a multiplication too: $A=M_g$, where $g=Ae\in\mathbf{A}_{0\infty}$.

For every $\tau\in[0,1]$, define C_τ by $C_\tau f(z,t)=f[z,1/2+(t-\tau)/2]$. By using, e.g., [1], it is not difficult to see that C_τ is bounded and commutes with A; therefore, $g(z, \tau)=Ae(z, \tau)=AC_\tau e(z, \tau)=C_\tau Ae(z, \tau)=C_\tau g(z, \tau)=g(z, 1/2)$, i.e., g depends only on z.

By the definition and properties of $\mathbf{A}_{0\infty}$, it follows that $g(z, t)$ is the norm-limit of a sequence of rational functions with poles outside Ω^-. Since $\mathscr{A}'(L_0)$ is strictly cyclic, this implies that $A=M_g$ is a norm-limit of rational functions of L_0 with poles outside Ω^- (see [37]). This proves (v).

(vi) It is obvious that for every $C\in\mathscr{L}(\mathfrak{R})$, appr $(C)''$ is inverse-closed, so that appr $(C)''$ always contains the norm-closure of the rational functions of C with poles outside $\sigma(C)$. Now (vi) follows from (v). □

Lemma 6. *Let Ω be an open set with analytic boundary, let n be a natural number, let $\mathbf{W}_{0n}$ be the direct sum of n copies of $\mathbf{W}^{2,1}(\partial\Omega, dm)$ and let $\mathbf{A}_{0n}$ be the subspace of "analytic elements" of $\mathbf{W}_{0n}$. Then $\mathbf{W}_{0n}$ and $\mathbf{A}_{0n}$ are Banach algebras with identity (under an equivalent norm), $\mathscr{M}(\mathbf{W}_{0n})\approx\partial\Omega\times\{1/n, 2/n, \dots, 1\}$ and $\mathscr{M}(\mathbf{A}_{0n})\approx$ $\approx\Omega^-\times\{1/n, 2/n, \dots, 1\}$.*

If $T_{0n}=M_z$ in $\mathbf{W}_{0n}$, then $\mathbf{A}_{0n}$ is invariant under T_{0n} and its restriction $L_{0n}=T_{0n}|\mathbf{A}_{0n}$ satisfy

(i) $\sigma(T_{0n})=E_l(T_{0n})=E_r(T_{0n})=E_l(L_{0n})=E_r(L_{0n})=\partial\Omega$.

(ii) $\sigma(L_{0n})=\Omega^-$.

(iii) $\operatorname{Ker}(\lambda-L_{0n})=\{0\}$ *and* $\dim\operatorname{Ker}(\lambda-L_{0n})^*=n$ *(so that* $\operatorname{ind}(\lambda-L_{0n})=-n$*) for every $\lambda\in\Omega$.*

(iv) $\mathscr{A}'(L_{0n})\cong\mathbf{A}_{0n}^{(n\times n)}$ *is the algebra of all $n\times n$ operator matrices with entries in $\{M_g: g\in\mathbf{A}_{0n}\}$, so that $\bar\mu[\mathscr{A}'(L_{0n})]=1$.*

(v) $\mathscr{A}^a(L_{0n})=\mathscr{A}''(L_{0n})\cong\{M_g: g\in\mathbf{A}_{0n}\}$ *= norm-closure of the rational functions of L_{0n} with poles outside Ω^-.*

(vi) appr $(L_{0n})''=\mathscr{A}''(L_{0n})$.

The proof (that can be easily "modelled" on that of Lemma 5) is left to the reader.

Now it is clear that if $T\dot{=}F\oplus\{\bigoplus_1^m L(\Omega_k, n_k)\}\oplus\{\bigoplus_{j=1}^p L(\Omega_j^*, -n_j)^*\}$, where F is an operator acting on a finite dimensional space, $L(\Omega, n)$ is the operator defined by Lemma 5 (for $n=-\infty$) and by Lemma 6 (for $-\infty<n<0$) and $\{\sigma(F), \{\Omega_k^-\}_{k=1}^m, \{\Omega_j^-\}_{j=1}^p\}$ $(0\leqq m, p<\infty)$ is a family of pairwise disjoint compact sets, then appr $(T)''=\mathscr{A}''(T)=$ norm-closure of the rational functions of T with poles outside $\sigma(T)$.

A formal repetition of the proof of Theorem 3 shows that the operators in $\mathscr{L}(\mathfrak{R})$ that are similar to some T as above form a dense subset, whence we obtain

Corollary 4. *(K) is dense in $\mathscr{L}(\mathfrak{R})$.*

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